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Numerical Treatments for Nonlinear Integro-Fractional Differential Equations of Volterra-Hammerstein Type using Runge-Kutta Method with the aid of Finite Difference Approximation

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Abstract

This paper presents a numerical solution scheme for an important class of nonlinear integro-fractional differential equations in Volterra-Hammerstein type of all positive arbitrary orders which are less than or equal to one. In this approach the nonlinear integro-differential equations are expressed in terms of Caputo type fractional derivative. An approach based on the Runge-Kutta methods including second, third and fourth orders with the aid of finite difference approximation. Further, we suggested an algorithm for these methods and for each algorithm the Computer program in MatLab was written in general forms. Finally, several illustrative examples are presented to show the effectiveness and accuracy of this method.

Introduction

The idea of this work is to provide a numerical technique for solving a nonlinear Integro-Fractional Differential Equations (IFDE) of positive arbitrary all-orders are less than one for Volterra-Hammerstein (V-H) type with variable coefficients in the general form for all $t \in [a,\mathcal{T}]$:

$$ {}_a^C D_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} p_i(t)\, {}_a^C D_t^{\alpha_i} u(t) + p_0(t) u(t) = f(t) + \lambda \sum_{\ell=0}^{m} \int_a^t \mathcal{K}_\ell(t,s)\, \mathcal{H}_\ell\left(s\,, {}_a^C D_s^{\beta_\ell} u(s)\right) ds \qquad (1)$$

Under the following initial condition:

$$u(t_0 = a) = u_0 \qquad (2)$$

Where $u(t)$ is the unknown real function which is a solution of equation (1) under the initial condition (2), the functions $\mathcal{K}_\ell \in C(S,\mathbb{R})$ and $\mathcal{H}_\ell \in C(S_* \times \mathbb{R},\mathbb{R})$; with $(S = \{(t,s): a \le s \le t \le \mathcal{T}\}$; $(S_* = \{s: a \le s \le t\,; t \le \mathcal{T}\})$; $\ell = 0,1,\dots,m$ and $f, p_i: [a,\mathcal{T}] \to \mathbb{R}$; $i = 0,1,2,\cdots,n$ are all real continuous functions. In addition, $\alpha_i\,, \beta_\ell \in \mathbb{R}^+$, $0 < \alpha_i, \beta_\ell \le 1$ for all $i = \overline{1{:}n}$ and $\ell = \overline{1{:}m}$ with property that: $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 =$

0; $\beta_m > \beta_{m-1} > \cdots > \beta_1 > \beta_0 = 0$ and λ is a scalar parameter. ${}_{a}^{C}D_t^{\gamma}$ Denotes the Caputo fractional derivative of order $\gamma \in \mathbb{R}^+, \gamma \leq 1$.
Our interest in equations (1-2) arises from its applications as a model for physical phenomena and bioengineering ***[8, 10, 20]***. Volterra-Hammerstein is one of important types, which is generalization of cases arrives in various branches of applications such as free electron laser and heat conduction in materials with memory ***[9, 18]***.

Actually most nonlinear integro-fractional differential equations (IFDEs) of Volterra-Hammerstein (V-H) type with variable coefficients do not have exact analytic solutions, therefore approximate and numerical techniques must be used. One of the popular methods for solving differential and integral equations is Runge-Kutta method. The origins of Runge-Kutta methods are used to find the numerical solution of initial value problems ***[5,7, 12, 21]***, and many researchers uses it in various orders to approximate the solution of integral and integro-differential equations ***[1, 3, 16]***. Moreover, Ahmed ***[2]*** applied it to solve numerically system of linear VIFDEs.

In this study, we discuss the numerical solution of equation (1) by good algorithms based on the Runge-Kutta method including second, third and fourth order ***[4, 15]***, finite difference approximation ***[2]*** and Caputo fractional derivative ***[22].*** Finally, computer package programs are written in MatLab for above algorithms.

Basic Preliminaries:

For the convenience of the reader, we present here the necessary information's from fractional calculus and auxiliary facts of R-K methods. This information's can be found in the resent literature.

A. Fractional Calculus [13,17,19,20]

Fractional calculus deals with the differentiation and integration of arbitrary order, many definitions of it have been proposed. We adopt Caputo's definition, which is a modification of the other definitions and has the advantage of dealing properly with initial value problem, for the concept of the fractional derivative. Here we discuss the definition and important properties of such a type of fractional derivative, which are used throughout this paper. We begin by defining the function space $C_\gamma, \gamma \in \mathbb{R}$, which was used in development of the operational calculus for the differential operator.

Definition 2.1: [19]
A real valued function u defined on $[a, b]$ be in the space $C_\gamma[a, b]$, $\gamma \in \mathbb{R}$, if there exist a real number $p > \gamma$, such that $u(t) = (t-a)^p u_*(t)$, where $u_* \in C[a, b]$, and it is said to be in the space $C_\gamma^n[a, b]$ if and only if $u^{(n)} \in C_\gamma[a, b], n \in \mathbb{N}_0$.

Definition 2.2: [13, 20]
Let $u \in C_\gamma[a, b], \gamma \geq -1$ and $\alpha \in \mathbb{R}^+$. Then the Riemann-Liouville fractional integral operator ${}_aJ_t^{\alpha}$of order α of a function u, is defined as:

$$ {}_aJ_t^{\alpha} u(t) = \int_a^t \frac{(t-\xi)^{\alpha-1} u(\xi)}{\Gamma(\alpha)} d\xi \ , \alpha > 0 $$

and when $\alpha = 0$, we have ${}_aJ_t^0 u(t) = u(t)$.

Definition 2.3: [20, 22]
The Caputo fractional derivative operator ${}_a^C D_t^{\alpha}$ of order $\alpha \in \mathbb{R}^+$ of a function $u \in C_{-1}^m[a, b]$ and $m - 1 < \alpha \leq m, m \in \mathbb{N}$ is defined as:

$$ {}_a^C D_t^{\alpha} u(t) = {}_aJ_t^{m-\alpha} D_t^m u(t) $$

Thus for $\alpha = m, m \in \mathbb{N}_0$, and $u \in C^m[a, b]$, we have for all $\leq t \leq b$; ${}_a^C D_t^0 u(t) = u(t)$; ${}_a^C D_t^m u(t) = D_t^m u(t)$

Note that: ***[13, 17, 20, 22]***

i. For $\alpha \geq 0$ and $\beta > 0$, then ${}_aJ_t^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1}$.
ii. For all $\alpha \geq 0$, $\beta \geq 0$ and $u(t) \in C_\gamma[a,b], \gamma \geq -1$, then: ${}_aJ_t^\alpha\, {}_aJ_t^\beta u(t) = {}_aJ_t^\beta\, {}_aJ_t^\alpha u(t) = {}_aJ_t^{\alpha+\beta} u(t)$
iii. ${}_a^CD_t^\alpha \mathcal{C} = 0$; $\mathcal{C}$ is any constant; ($\alpha \geq 0, \alpha \notin \mathbb{N}$)
iv. Assume that $u \in C_{-1}^m[a,b]$; $\alpha \geq 0, \alpha \notin \mathbb{N}$ and $m = \lceil \alpha \rceil$ then ${}_a^CD_t^\alpha u(t)$ is continuous on $[a,b]$, and $[{}_a^CD_t^\alpha u(t)\,]_{t=a} = 0$.
v. Let $\alpha \geq 0$, $m = \lceil \alpha \rceil$ and $u \in C^m[a,b]$, then, the relation between the Caputo derivative and R-L integral are formed:

$${}_a^CD_t^\alpha\, {}_aJ_t^\alpha u(t) = u(t) \;;\quad a \leq t \leq b \qquad\qquad {}_aJ_t^\alpha\, {}_a^CD_t^\alpha u(t) = u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(a)}{k!}(t-a)^k$$

The three basic propositions it is necessary to our work are presented here:

Proposition 1: ***[22]***

Suppose that $\alpha \geq 0$; $m = \lceil \alpha \rceil$ and the α-Caputo derivative for power function $u(t) = (t-a)^\beta$ for some $\beta \geq 0$, formulate as:

$${}_a^CD_t^\alpha u(t) = \begin{cases} 0 & if\ \beta \in \{0,1,2,\cdots,m-1\} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\beta-\alpha} & \begin{array}{l} if\ \beta \in \mathbb{N}\ and\ \beta \geq m \\ or\ \beta \notin \mathbb{N}\ and\ \beta > m-1 \end{array} \end{cases}$$

B. Auxiliary Facts [4, 6, 14, 15]

Classes of algorithms called explicit Runge-Kutta methods is popular in practice and have been studied in profoundly. The origins of R-K methods for the approximate solution at the points $t_r = a + rh\,, r = 1,2,\dots,N$ of the initial value problem:

$$\frac{du}{dt} = f\big(t, u(t)\big) \;;\; u(t_0) = u_0$$

Depend on generating some intermediate points in $[t_r, t_{r+1}]$; $r = \overline{0{:}\,N-1}$ these points are symbolized by $t_r + \theta_v h$; $v = 1,2,\dots,p-1$ ($p > 1$; $p \in \mathbb{Z}^+$) and $0 = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{p-1} = 1$. The general p-stage Runge-Kutta methods for the above initial value problem is defined by:

$$u_{r+1} = u_r + h\sum_{\ell=0}^{p-1} A_{p\ell} f\big(t_r + \theta_\ell h, u_{r+\theta_\ell}\big) \tag{3}$$

Where

$$\sum_{\ell=0}^{v-1} A_{v\ell} = \begin{cases} \theta_v & , \quad v = 1,2,\dots,p-1 \\ 1 & , \quad v = p \end{cases}$$

The parameters $A_{p\ell}$ and θ_ℓ are chosen in practice to yield a final approximation of specified order ***[11]***. In the preceding method we have expressed u_{r+1} in terms of previously calculated.

Before starting the investigations, it is necessary to give some basic two propositions that are proved in ***[2]*** which are basic in our works:

Proposition 2:

The finite difference approximation of Caputo derivative for $0 < \alpha \leq 1$ at defines points $t = t_{r+1}$; $r = 0,1,\dots,N-1$ and $h = (b-a)/N$, is formed as

$${}_a^CD_t^\alpha u(t_{r+1}) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{r}\big[u(t_{r-j+1}) - u(t_{r-j})\big]\, b_j^\alpha$$

where $b_j^\alpha = (j+1)^{1-\alpha} - j^{1-\alpha}$.

Proposition 3:

For any p-stage of R-K methods with θ_j's ($j = 0,1,\dots,p-1$) such that $0 = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{p-1} = 1$. The Caputo fractional derivative of order $0 < \alpha < 1$ for any smooth function $u(t)$ on $[a,b]$, can evaluate it at any points $t = t_r + \theta_v h$ for each $r = 0,1,\dots,N-1$ with $v = 1,2,\dots,p-1$ by formula:

$$\left[{}_{a}^{C}D_{t}^{\alpha}u(t)\right]_{t=t_r+\theta_v h}$$
$$= \frac{[h/(p-1)]^{-\alpha}}{\Gamma(2-\alpha)}\left\{\sum_{j=0}^{r-1}\sum_{\ell=0}^{p-2}\left[u\left(t_j+\theta_{\ell+1}h\right)-u\left(t_j+\theta_{\ell}h\right)\right]b_{(p-1)(r-j)+(v-\ell)-1}^{\alpha} + \sum_{j=0}^{v-1}\left[u\left(t_r+\theta_{j+1}h\right)-u\left(t_r+\theta_j h\right)\right]b_{v-j-1}^{\alpha}\right\}$$

where $b_j^{\alpha} = (j+1)^{1-\alpha} - j^{1-\alpha}$, for all positive integer numbers j and positive real numbers α.

Solution Technique:

In this section, some new algorithms are presented for treating nonlinear IFDEs of V-H type (1-2) using Runge-Kutta method of various order with aid of finite difference approximation. Recall equations (1 to 2) for $0 < \max\{\lceil\alpha_n\rceil, \lceil\beta_m\rceil\} \leq 1$ and setting $t = t_r = a + rh$; $h = \frac{b-a}{N}$; $(N \geq 1)$; for all $r = 1,2,\dots,N$ with take $p_n(t) = 1$, that is

$$\sum_{i=1}^{n} p_i(t_r)\left[{}_{a}^{C}D_{t}^{\alpha_i}u(t)\right]_{t=t_r} + p_0(t_r)u(t_r) = f(t_r) + \lambda\sum_{\ell=0}^{m}\sum_{j=0}^{r-1}\int_{a+jh}^{a+(j+1)h}\mathcal{K}_{\ell}(a+rh,s)\,\mathcal{H}_{\ell}\left(s\,,{}_{a}^{C}D_{s}^{\beta_\ell}u(s)\right)ds$$

Now, by applying proposition (2) for left hand side and equation (3) for above integral part, we obtain the following equation to determining an approximation of $u(t_r)$ for each $r = 1,2,\cdots,N$:

$$\sum_{i=1}^{n} p_i(t_r)\frac{h^{-\alpha_i}}{\Gamma(2-\alpha_i)}\left[\sum_{k=0}^{r-1}[u(t_{r-k}) - u(t_{r-k-1})]b_k^{\alpha_i}\right] + p_0(t_r)u(t_r)$$
$$= f(t_r) + \lambda\sum_{\ell=0}^{m}\sum_{j=0}^{r-1} h\left[\sum_{L=0}^{p-1} A_{pL}\mathcal{K}_{\ell}(a+rh, a+(j+\theta_L)h)\mathcal{H}_{\ell}\left(a+(j+\theta_L)h\,,\left[{}_{a}^{C}D_{s}^{\beta_\ell}u(s)\right]_{s=t_j+\theta_L h}\right)\right] \quad (4)$$

Applying proposition (3) to evaluate β_ℓ-Caputo fractional part at each points $t_j + \theta_L h$, we get equation (5) as:

$$\left[{}_{a}^{C}D_{s}^{\beta_\ell}u(s)\right]_{s=t_j+\theta_L h}$$
$$= \frac{[h/(p-1)]^{-\beta_\ell}}{\Gamma(2-\beta_\ell)}\left[\sum_{i=0}^{j-1}\sum_{k=0}^{p-2}[u(t_i+\theta_{k+1}h) - u(t_i+\theta_k h)]\,b_{(p-1)(j-i)+(L-k)-1}^{\beta_\ell} + \sum_{i=0}^{L-1}\left[u\left(t_j+\theta_{i+1}h\right)-u\left(t_j+\theta_i h\right)\right]b_{L-i-1}^{\beta_\ell}\right] \quad (5)$$

Now, for $t \in (t_r, t_{r+1})$ we may write equation (1) in the following form:

$$\sum_{i=1}^{n} p_i(t)\,{}_{a}^{C}D_{t}^{\alpha_i}u(t) + p_0(t)u(t)$$
$$= f(t) + \lambda\sum_{\ell=0}^{m}\left[\sum_{k=0}^{r-1}\int_{t_k}^{t_{k+1}}\mathcal{K}_{\ell}(t,s)\,\mathcal{H}_{\ell}\left(s\,,{}_{a}^{C}D_{s}^{\beta_\ell}u(s)\right)ds + \int_{t_r}^{t}\mathcal{K}_{\ell}(t,s)\,\mathcal{H}_{\ell}\left(s\,,{}_{a}^{C}D_{s}^{\beta_\ell}u(s)\right)ds\right] \quad (6)$$

Then setting $t = t_r + \theta_v h$; $v = 1,2,\dots,p-1$; and using proposition (3) for left hand side of equation (6), then approximating the final integral in above equation by:

$$\int_{t_r}^{t_r+\theta_v h}\mathcal{K}_{\ell}(t_r+\theta_v h, s)\,\mathcal{H}_{\ell}\left(s\,,{}_{a}^{C}D_{s}^{\beta_\ell}u(s)\right)ds$$
$$\cong h\sum_{L=0}^{v-1} A_{vL}\mathcal{K}_{\ell}(t_r+\theta_v h\,, t_r+\theta_L h)\mathcal{H}_{\ell}\left(t_r+\theta_L h, \left[{}_{a}^{C}D_{s}^{\beta_\ell}u(s)\right]_{s=t_r+\theta_L h}\right)$$

So, the Runge-Kutta method for nonlinear IFDE of V-H type of the form (1) may be expressed as:

$$\sum_{i=1}^{n} p_i(t_r+\theta_v h)\frac{[h/(p-1)]^{-\alpha_i}}{\Gamma(2-\alpha_i)}\left[\sum_{j=0}^{r-1}\sum_{\ell=0}^{p-2}\left[u\big(t_j+\theta_{\ell+1}h\big)-u\big(t_j+\theta_\ell h\big)\right]b^{\alpha_i}_{(p-1)(r-j)+(v-\ell)-1}\right.$$
$$\left.+\sum_{j=0}^{v-1}\left[u\big(t_r+\theta_{j+1}h\big)-u\big(t_r+\theta_j h\big)\right]b^{\alpha_i}_{v-j-1}\right]+p_0(t_r+\theta_v h)u(t_r+\theta_v h)$$
$$=f(t_r+\theta_v h)+\lambda h\sum_{\ell=0}^{m}\left[\sum_{k=0}^{r-1}\sum_{L=0}^{p-1}A_{pL}\mathcal{K}_\ell(t_r+\theta_v h,t_k+\theta_L h)\mathcal{H}_\ell\left(t_k+\theta_L h,\left[{}^{C}_{a}D^{\beta_\ell}_{s}u(s)\right]_{s=t_k+\theta_L h}\right)\right.$$
$$\left.+\sum_{L=0}^{v-1}A_{vL}\mathcal{K}_\ell(t_r+\theta_v h,t_r+\theta_L h)\,\mathcal{H}_\ell\left(t_r+\theta_L h\,,\left[{}^{C}_{a}D^{\beta_\ell}_{s}u(s)\right]_{s=t_r+\theta_L h}\right)\right] \qquad (7)$$

For all $v=1,2,\dots,p-1$; and $u(t_0)=u_0$ (is given). For suitable choice to the parameters A_{pL} and $\theta_\ell$in equation (7), the following particular methods are given:

A. Second order Runge-Kutta method:

Here, we take the order of Runge-Kutta method $p=2$, and we have the following parameters ***[2, 11]:***

$\theta_0=0$, $\theta_1=1$, $A_{10}=1$, and $A_{20}=A_{21}=\frac{1}{2}$

Now, putting above parameters in equation (7) with noting that v has only one value which is one, and for all $r=0,1,\dots,N-1$; we obtain the following formula:

$$\sum_{i=1}^{n} p_i(t_r+h)\frac{h^{-\alpha_i}}{\Gamma(2-\alpha_i)}\left\{\sum_{j=0}^{r-1}\left[u\big(t_j+h\big)-u\big(t_j\big)\right]b^{\alpha_i}_{(r-j)}+\left[u(t_r+h)-u(t_r)\right]\right\}+p_0(t_r+h)u(t_r+h)$$
$$=f(t_r+h)$$
$$+\frac{\lambda h}{2}\sum_{\ell=0}^{m}\left\{\sum_{k=0}^{r-1}\left[\mathcal{K}_\ell(t_r+h,t_k)\,\mathcal{H}_\ell\left(t_k\,,\left[{}^{C}_{a}D^{\beta_\ell}_{s}u(s)\right]_{s=t_k}\right)\right.\right.$$
$$\left.+\mathcal{K}_\ell(t_r+h,t_k+h)\,\mathcal{H}_\ell\left(t_k+h\,,\left[{}^{C}_{a}D^{\beta_\ell}_{s}u(s)\right]_{s=t_k+h}\right)\right]$$
$$\left.+2\mathcal{K}_\ell(t_r+h,t_r)\,\mathcal{H}_\ell\left(t_r\,,\left[{}^{C}_{a}D^{\beta_\ell}_{s}u(s)\right]_{s=t_r}\right)\right\} \qquad (8)$$

where $b^{\alpha}_{j}=(j+1)^{1-\alpha}-j^{1-\alpha}$. Now, suppose for each $r=0,1,\dots,N-1$ and $i=0,1,\dots,n$. we symbolize:

$$C^r_i=p_i(t_r+h)\frac{h^{-\alpha_i}}{\Gamma(2-\alpha_i)} \qquad (9)$$

$$\mathcal{A}_r=\sum_{i=1}^{n}C^r_i \qquad (10)$$

For easy to use we will denote $\mathcal{D}^{\ell}_{\xi}=\left[{}^{C}_{a}D^{\beta_\ell}_{s}u(s)\right]_{s=t_\xi}$; then for all $r=0,1,\dots,N-1$ equation (8) equivalent to the following formula:

$$(\mathcal{A}_r + C_0^r)\, u(t_{r+1})$$
$$= \mathcal{A}_r\, u(t_r) - \sum_{j=0}^{r-1}\left[u(t_{j+1}) - u(t_j)\right]\left(\sum_{i=1}^{n} C_i^r \;\; b_{r-j}^{\alpha_i}\right) + f(t_{r+1})$$
$$+ \frac{\lambda h}{2}\sum_{\ell=0}^{m}\sum_{k=0}^{r-1}\{\mathcal{K}_\ell(t_{r+1}, t_k)\, \mathcal{H}_\ell(t_k\,, \mathcal{D}_k^\ell) + \mathcal{K}_\ell(t_{r+1}, t_{k+1})\, \mathcal{H}_\ell(t_{k+1}\,, \mathcal{D}_{k+1}^\ell)\}$$
$$+ \lambda h \sum_{\ell=0}^{m} \mathcal{K}_\ell(t_{r+1}, t_r)\, \mathcal{H}_\ell(t_r\,, \mathcal{D}_r^\ell) \qquad (11)$$

With given starting values from the initial condition $u(t_0) = u_0$. Furthermore, the formula (11) is explicit multistep formulas.

B. Algorithm of Two Order Runge-Kutta Method (ARKM2)

The approximate solution of multi-order nonlinear IFDE of V-H type with variable coefficients (1-2) by using second order Runge-kutta method with the aid of finite difference approximation can be summarized by the following steps:

Step1:

i- Let $h = \frac{b-a}{N}$; $N \in \mathbb{N}$ and $t_k = a + kh \quad (k = \overline{0:N}\,)$.

ii- Set $u_0 = u(t_0)$ [given from equation (2)].

Step2: For all $r = 0,1, \dots, N-1$:

i- Evaluate C_i^r for each $i = 0,1, \dots, n$. by equation (9).

ii- Compute $\mathcal{A}_r$ by applying equation (10).

Step3: For each $\ell = 0,1, \dots, m$ and $k = 0,1, \dots, r$ calculate: $\mathcal{D}[\ell, k] = \left[{}_a^C D_s^{\beta_\ell} u(s)\right]_{s=t_k}$ from the proposition 2.

Step4: For each $r = 0,1, \dots, N-1$using step (2, i) and step 3 to compute the following:

i. $\mathcal{B}[r] = -\sum_{j=0}^{r-1}\left[u(t_{j+1}) - u(t_j)\right]\left(\sum_{i=1}^{n} C_i^r \;\; b_{r-j}^{\alpha_i}\right)$ with $\mathcal{B}[0] = 0$

ii. $\mathcal{R}[r] = \lambda h \sum_{\ell=0}^{m} \mathcal{K}_\ell(t_{r+1}, t_r)\, \mathcal{H}_\ell(t_r\,, \mathcal{D}[\ell, r])$

Step5: Compute $u(t_{r+1})$ for each $r = 0,1, \dots, N-1$ from:

$$(\mathcal{A}_r + C_0^r)\, u(t_{r+1}) = \mathcal{A}_r\, u(t_r) + \mathcal{F}(t_{r+1}) + \mathcal{B}[r] + \mathcal{R}[r].$$

where

$$\mathcal{F}(t) = f(t) + \frac{\lambda h}{2}\sum_{\ell=0}^{m}\sum_{k=0}^{r-1}\{\mathcal{K}_\ell(t, t_k)\, \mathcal{H}_\ell(t_k\,, \mathcal{D}[\ell, k]) + \mathcal{K}_\ell(t, t_{k+1})\, \mathcal{H}_\ell(t_{k+1}\,, \mathcal{D}[\ell, k+1])\}$$

with putting $\quad b_\xi^\alpha = (\xi + 1)^{1-\alpha} - \xi^{1-\alpha}$; $\xi = 0,1,2, \cdots$

C. Third order Runge-Kutta method:

Here, we take the order of Runge-Kutta method $p = 3$, and we have the following parameters [2, 11]:

$$\theta_0 = 0 \;, \theta_1 = \frac{1}{2} \,, \theta_2 = 1 \,, \text{ and}$$
$$A_{10} = \frac{1}{2} \,, A_{20} = -1 \,, A_{21} = 2, A_{31} = \frac{2}{3} \,;\, A_{30} = A_{32} = \frac{1}{6} \quad .$$

Now, putting above parameters in equation (7) with noting that v have two values 1 and 2. For easy to work we denote $\mathcal{D}_\xi^\ell = \left[{}_a^C D_s^{\beta_\ell} u(s)\right]_{s=t_\xi}$ and using proposition (2), (3) respectively; for all $r = 0,1, \dots, N-1$ we obtain the following two iterated equations for $\boldsymbol{v}$ equal to **1** and **2** respectively:

For $v = 1$

$$\left\{\mathcal{A}_r^{h/2} + C_0^r\left(\frac{h}{2}\right)\right\} u(t_r + \theta_1 h)$$
$$= \mathcal{A}_r^{h/2}\, u(t_r)$$
$$- \sum_{j=0}^{r-1}\left\{\left[u\left(t_j + \theta_1 h\right) - u\left(t_j\right)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right)\, b_{2(r-j)}^{\alpha_i}\right)\right.$$
$$\left. + \left[u\left(t_{j+1}\right) - u\left(t_j + \theta_1 h\right)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right)\, b_{2(r-j)-1}^{\alpha_i}\right)\right\} + f\left(t_r + \frac{h}{2}\right)$$
$$+ \frac{\lambda h}{6}\sum_{\ell=0}^{m}\sum_{k=0}^{r-1}\left\{\mathcal{K}_\ell\left(t_r + \frac{h}{2}, t_k\right)\mathcal{H}_\ell\left(t_k, \mathcal{D}_k^\ell\right) + 4\mathcal{K}_\ell\left(t_r + \frac{h}{2}, t_k + \frac{h}{2}\right)\mathcal{H}_\ell\left(t_k + \frac{h}{2}, \mathcal{D}_{k+\theta_1}^\ell\right)\right.$$
$$\left. + \mathcal{K}_\ell\left(t_r + \frac{h}{2}, t_k + h\right)\mathcal{H}_\ell\left(t_{k+1}, \mathcal{D}_{k+1}^\ell\right)\right\} + \frac{\lambda h}{2}\sum_{\ell=0}^{m}\mathcal{K}_\ell\left(t_r + \frac{h}{2}, t_r\right)\mathcal{H}_\ell\left(t_r, \mathcal{D}_r^\ell\right) \qquad (12)$$

For $v = 2$

$$\{\mathcal{A}_r^h + C_0^r(h)\}\, u(t_{r+1})$$
$$= \mathcal{A}_r^h\, u(t_r + \theta_1 h) - [u(t_r + \theta_1 h) - u(t_r)]\left(\sum_{i=1}^{n} C_i^r(h)\; b_1^{\alpha_i}\right)$$
$$- \sum_{j=0}^{r-1}\left\{\left[u\left(t_j + \theta_1 h\right) - u\left(t_j\right)\right]\left(\sum_{i=1}^{n} C_i^r(h)\; b_{2(r-j)+1}^{\alpha_i}\right)\right.$$
$$\left. + \left[u\left(t_{j+1}\right) - u\left(t_j + \theta_1 h\right)\right]\left(\sum_{i=1}^{n} C_i^r(h)\; b_{2(r-j)}^{\alpha_i}\right)\right\} + f(t_{r+1})$$
$$+ \frac{\lambda h}{6}\sum_{\ell=0}^{m}\sum_{k=0}^{r-1}\left\{\mathcal{K}_\ell(t_{r+1}, t_k)\,\mathcal{H}_\ell\left(t_k, \mathcal{D}_k^\ell\right) + 4\mathcal{K}_\ell\left(t_r + h, t_k + \frac{h}{2}\right)\mathcal{H}_\ell\left(t_k + \frac{h}{2}, \mathcal{D}_{k+\theta_1}^\ell\right)\right.$$
$$\left. + \mathcal{K}_\ell(t_{r+1}, t_k + h)\,\mathcal{H}_\ell\left(t_{k+1}, \mathcal{D}_{k+1}^\ell\right)\right\}$$
$$- \lambda h\sum_{\ell=0}^{m}\left\{\mathcal{K}_\ell(t_{r+1}, t_r)\,\mathcal{H}_\ell\left(t_r, \mathcal{D}_r^\ell\right) - 2\mathcal{K}_\ell\left(t_{r+1}, t_r + \frac{h}{2}\right)\mathcal{H}_\ell\left(t_r + \frac{h}{2}, \mathcal{D}_{r+\theta_1}^\ell\right)\right\} \qquad (13)$$

where $b_j^\alpha = (j+1)^{1-\alpha} - j^{1-\alpha}$. Let, for each $r = 0,1,\dots,N-1$; $z = h \; and \; h/2$ and $i = \overline{0\!:\!n}$.

$$C_i^r(z) = p_i(t_r + z)\frac{[h/2]^{-\alpha_i}}{\Gamma(2 - \alpha_i)} \qquad (14)$$

$$\mathcal{A}_r^z = \sum_{i=1}^{n} C_i^r(z) \qquad (15)$$

With given starting values from the initial condition $u(t_0) = u_0$. Furthermore, the formulas (12 and 13) are explicit multistep formulas.

D. Algorithm of Third Order Runge-Kutta Method (ARKM3)

The approximate solution of multi-order nonlinear IFDE of V-H type with variable coefficients (1-2) by using third order Runge-kutta method with the aid of finite difference approximation can be summarized by the following stages:

Step 1:

i- Let $h = \frac{b-a}{N}$; $N \in \mathbb{N}$ and $t_k = a + kh$ $(k = \overline{0\!:\!N})$.

ii- Set $u_0 = u(t_0)$ [given from equation (2)].

Step 2: For all $r = 0,1,\dots,N-1$ with $z = \frac{h}{2}$, h:

i- Evaluate $C_i^r(z)$ for each $i = 0,1,\dots,n$. by equation (14).

ii- Compute $\mathcal{A}_r^z$ by equation (15).

Step 3: For each $\ell = 0,1,\dots,m$ and $k = 0,1,\dots,r$ calculate:

$\mathcal{D}[\ell,k] = \left[{}_a^C D_s^{\beta_\ell} u(s)\right]_{s=t_k}$ and $\mathcal{D}[\ell,k+\theta_1] = \left[{}_a^C D_s^{\beta_\ell} u(s)\right]_{s=t_{k+\theta_1}}$ from the proposition (2 and 3).

Step 4: For each $r = 0,1,\dots,N-1$ using step (2, i) and step (3) to compute the following:

$$\text{i. } \mathcal{B}_1[r] = -\sum_{j=0}^{r-1}\left\{\left[u(t_j+\theta_1 h) - u(t_j)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_{2(r-j)}^{\alpha_i}\right) + \left[u(t_{j+1}) - u(t_j+\theta_1 h)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_{2(r-j)-1}^{\alpha_i}\right)\right\} \text{ with } \mathcal{B}_1[0] = 0$$

$$\text{ii. } \mathcal{R}_1[r] = \frac{\lambda h}{2}\sum_{\ell=0}^{m}\mathcal{K}_\ell\left(t_r+\frac{h}{2}, t_r\right)\mathcal{H}_\ell(t_r, \mathcal{D}[\ell,r])$$

Step 5: Calculate for each $r = 0,1,\dots,N-1$

$$\text{i. } u(t_r+\theta_1 h) = \frac{1}{\mathcal{A}_r^{h/2} + C_0^r\left(\frac{h}{2}\right)}\left\{\mathcal{A}_r^{h/2}u(t_r) + \mathcal{B}_1[r] + \mathcal{R}_1[r] + \mathcal{F}(t_r+\frac{h}{2})\right\}.$$

$$\text{ii. } \mathcal{B}_2[r] = -[u(t_r+\theta_1 h) - u(t_r)]\left(\sum_{i=1}^{n} C_i^r(h)\, b_1^{\alpha_i}\right) - \sum_{j=0}^{r-1}\left\{\left[u(t_j+\theta_1 h) - u(t_j)\right]\left(\sum_{i=1}^{n} C_i^r(h)\, b_{2(r-j)+1}^{\alpha_i}\right) + \left[u(t_j+h) - u(t_j+\theta_1 h)\right]\left(\sum_{i=1}^{n} C_i^r(h)\, b_{2(r-j)}^{\alpha_i}\right)\right\}$$

$$\text{iii. } \mathcal{R}_2[r] = -\lambda h\sum_{\ell=0}^{m}\left\{\mathcal{K}_\ell(t_{r+1}, t_r)\,\mathcal{H}_\ell(t_r, \mathcal{D}[\ell,r]) - 2\mathcal{K}_\ell\left(t_{r+1}, t_r+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_r+\frac{h}{2}, \mathcal{D}[\ell, r+\theta_1]\right)\right\}$$

Step 6: Compute $u_{r+1} = u(t_{r+1})$ for each $r = 0,1,\dots,N-1$ from:

$$\{\mathcal{A}_r^h + C_0^r(h)\}\, u(t_{r+1}) = \mathcal{A}_r^h\, u(t_r+\theta_1 h) + \mathcal{B}_2[r] + \mathcal{R}_2[r] + \mathcal{F}(t_r+h)$$

where

$$\mathcal{F}(t) = f(t) + \frac{\lambda h}{6}\sum_{\ell=0}^{m}\sum_{k=0}^{r-1}\left\{\mathcal{K}_\ell(t,t_k)\,\mathcal{H}_\ell(t_k, \mathcal{D}[\ell,k]) + 4\mathcal{K}_\ell\left(t, t_k+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_k+\frac{h}{2}, \mathcal{D}[\ell, k+\theta_1]\right) + \mathcal{K}_\ell(t, t_k+h)\,\mathcal{H}_\ell(t_k+h, \mathcal{D}[\ell,k+1])\right\}$$

with putting $b_\xi^\alpha = (\xi+1)^{1-\alpha} - \xi^{1-\alpha}$; $\xi = 0,1,2,\cdots$

E. Fourth order Runge-Kutta method:

Here, we take the order of Runge-Kutta method $p = 4$, and we have the following parameters [2, 11]:

$$\theta_0 = 0\ , \theta_1 = \theta_2 = \frac{1}{2}, \theta_3 = 1,\ A_{10} = \frac{1}{2}\ , A_{20} = 0\ , A_{21} = \frac{1}{2}$$

$$A_{30} = A_{31} = 0, A_{32} = 1, A_{40} = A_{43} = \frac{1}{6}, A_{41} = A_{42} = \frac{1}{3}\ .$$

Now, putting above parameters in equation (7) with noting that v have three values 1,2 and 3. For easy to work we denote $\mathcal{D}_{\xi}^{\ell}=\left[{}_{a}^{C}D_{s}^{\beta_{\ell}}u(s)\right]_{s=t_{\xi}}$ and using propositions (2) and (3) respectively. Thus; for all $r=0,1,\dots,N-1$ we obtain the following three iterated equations for $\boldsymbol{v}=\mathbf{1},\mathbf{2}$ and $\mathbf{3}$ respectively:

For $\boldsymbol{v}=\mathbf{1}$

$$\begin{aligned}
&\left\{\mathcal{A}_r^{\theta_1 h}+C_0^r(\theta_1 h)\right\}\,u(t_r+\theta_1 h)\\
&\qquad=\mathcal{A}_r^{\theta_1 h}\,u(t_r)\\
&\qquad-\sum_{j=0}^{r-1}\left\{\left[u\left(t_j+\theta_1 h\right)-u\left(t_j\right)\right]\left(\sum_{i=1}^{n}C_i^r\left(\frac{h}{2}\right)\,b_{3(r-j)}^{\alpha_i}\right)\right.\\
&\qquad+\left[u\left(t_j+\theta_2 h\right)-u\left(t_j+\theta_1 h\right)\right]\left(\sum_{i=1}^{n}C_i^r\left(\frac{h}{2}\right)\,b_{3(r-j)-1}^{\alpha_i}\right)\\
&\qquad\left.+\left[u\left(t_{j+1}\right)-u\left(t_j+\theta_2 h\right)\right]\left(\sum_{i=1}^{n}C_i^r\left(\frac{h}{2}\right)\,b_{3(r-j)-2}^{\alpha_i}\right)\right\}+f\left(t_r+\frac{h}{2}\right)\\
&\qquad+\frac{\lambda h}{6}\sum_{\ell=0}^{m}\sum_{k=0}^{r-1}\left\{\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_k\right)\mathcal{H}_\ell\left(t_k,\mathcal{D}_k^{\ell}\right)+2\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_k+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_k+\frac{h}{2},\mathcal{D}_{k+\theta_1}^{\ell}\right)\right.\\
&\qquad\left.+2\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_k+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_k+\frac{h}{2},\mathcal{D}_{k+\theta_2}^{\ell}\right)+\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_{k+1}\right)\mathcal{H}_\ell\left(t_{k+1},\mathcal{D}_{k+1}^{\ell}\right)\right\}\\
&\qquad+\frac{\lambda h}{2}\sum_{\ell=0}^{m}\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_r\right)\mathcal{H}_\ell\left(t_r,\mathcal{D}_r^{\ell}\right)
\end{aligned}\tag{16}$$

For $\boldsymbol{v}=\mathbf{2}$

$$\begin{aligned}
&\left\{\mathcal{A}_r^{\theta_2 h}+C_0^r(\theta_2 h)\right\}\,u(t_r+\theta_2 h)\\
&\qquad=\mathcal{A}_r^{\theta_2 h}\,u(t_r+\theta_1 h)\\
&\qquad-\sum_{j=0}^{r-1}\left\{\left[u\left(t_j+\theta_1 h\right)-u\left(t_j\right)\right]\left(\sum_{i=1}^{n}C_i^r\left(\frac{h}{2}\right)\,b_{3(r-j)+1}^{\alpha_i}\right)\right.\\
&\qquad+\left[u\left(t_j+\theta_2 h\right)-u\left(t_j+\theta_1 h\right)\right]\left(\sum_{i=1}^{n}C_i^r\left(\frac{h}{2}\right)\,b_{3(r-j)}^{\alpha_i}\right)\\
&\qquad\left.+\left[u\left(t_{j+1}\right)-u\left(t_j+\theta_2 h\right)\right]\left(\sum_{i=1}^{n}C_i^r\left(\frac{h}{2}\right)\,b_{3(r-j)-1}^{\alpha_i}\right)\right\}\\
&\qquad-\left[u(t_r+\theta_1 h)-u(t_r)\right]\left(\sum_{i=1}^{n}C_i^r\left(\frac{h}{2}\right)\,b_1^{\alpha_i}\right)+f\left(t_r+\frac{h}{2}\right)\\
&\qquad+\frac{\lambda h}{6}\sum_{\ell=0}^{m}\sum_{k=0}^{r-1}\left\{\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_k\right)\mathcal{H}_\ell\left(t_k,\mathcal{D}_k^{\ell}\right)+2\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_k+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_k+\frac{h}{2},\mathcal{D}_{k+\theta_1}^{\ell}\right)\right.\\
&\qquad\left.+2\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_k+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_k+\frac{h}{2},\mathcal{D}_{k+\theta_2}^{\ell}\right)+\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_{k+1}\right)\mathcal{H}_\ell\left(t_{k+1},\mathcal{D}_{k+1}^{\ell}\right)\right\}\\
&\qquad+\frac{\lambda h}{2}\sum_{\ell=0}^{m}\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_r+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_r+\frac{h}{2},\mathcal{D}_{r+\theta_1}^{\ell}\right)
\end{aligned}\tag{17}$$

For $v = 3$

$$
\begin{aligned}
&\{\mathcal{A}_r^h + C_0^r(h)\}\, u(t_{r+1}) \\
&\qquad = \mathcal{A}_r^h\, u(t_r + \theta_2 h) \\
&\qquad - \sum_{j=0}^{r-1} \Bigg\{ \big[u\big(t_j + \theta_1 h\big) - u\big(t_j\big)\big] \left(\sum_{i=1}^{n} C_i^r(h)\ b_{3(r-j)+2}^{\alpha_i} \right) \\
&\qquad + \big[u\big(t_j + \theta_2 h\big) - u\big(t_j + \theta_1 h\big)\big] \left(\sum_{i=1}^{n} C_i^r(h)\ b_{3(r-j)+1}^{\alpha_i} \right) \\
&\qquad + \big[u\big(t_{j+1}\big) - u\big(t_j + \theta_2 h\big)\big] \left(\sum_{i=1}^{n} C_i^r(h)\ b_{3(r-j)}^{\alpha_i} \right) \Bigg\} - [u(t_r + \theta_1 h) - u(t_r)] \left(\sum_{i=1}^{n} C_i^r(h)\ b_2^{\alpha_i} \right) \\
&\qquad - [u(t_r + \theta_2 h) - u(t_r + \theta_1 h)] \left(\sum_{i=1}^{n} C_i^r(h)\ b_1^{\alpha_i} \right) + f(t_{r+1}) \\
&\qquad + \frac{\lambda h}{6} \sum_{\ell=0}^{m} \sum_{k=0}^{r-1} \Bigg\{ \mathcal{K}_\ell(t_{r+1}, t_k)\, \mathcal{H}_\ell\big(t_k, \mathcal{D}_k^\ell\big) + 2\mathcal{K}_\ell\left(t_{r+1}, t_k + \frac{h}{2}\right) \mathcal{H}_\ell\left(t_k + \frac{h}{2}, \mathcal{D}_{k+\theta_1}^\ell\right) \\
&\qquad + 2\mathcal{K}_\ell\left(t_{r+1}, t_k + \frac{h}{2}\right) \mathcal{H}_\ell\left(t_k + \frac{h}{2}, \mathcal{D}_{k+\theta_2}^\ell\right) + \mathcal{K}_\ell(t_{r+1}, t_{k+1})\, \mathcal{H}_\ell\big(t_{k+1}, \mathcal{D}_{k+1}^\ell\big) \Bigg\} \\
&\qquad + \lambda h \sum_{\ell=0}^{m} \mathcal{K}_\ell\left(t_{r+1}, t_r + \frac{h}{2}\right) \mathcal{H}_\ell\left(t_r + \frac{h}{2}, \mathcal{D}_{r+\theta_2}^\ell\right) \qquad (18)
\end{aligned}
$$

where $b_j^\alpha = (j+1)^{1-\alpha} - j^{1-\alpha}$. for $0 < \alpha \le 1$,Let, for each $r = 0,1,\dots,N-1$ and $i = 0,1,\dots,n$; $z = h/2$ and h:

$$
C_i^r(z) = p_i(t_r + z)\frac{[h/3]^{-\alpha_i}}{\Gamma(2-\alpha_i)} \qquad (19)
$$

$$
\mathcal{A}_r^z = \sum_{i=1}^{n} C_i^r(z) \qquad (20)
$$

with given starting values from the initial condition $u(t_0) = u_0$. Furthermore, the formula (16, 17 and 18) are explicit multi-step formula.

F. Algorithm of Forth Order Runge-Kutta Method (ARKM4)

The approximate solution of multi-order nonlinear IFDE of V-H type with variable coefficients (1-2) by using fourth order Runge-kutta method with the aid of finite difference approximation can be summarized by the following stages:

Step 1:

i. Let $h = \frac{b-a}{N}$; $N \in \mathbb{N}$ and $t_k = a + kh$ $(k = \overline{0{:}N})$.

ii. Set $u_0 = u(t_0)$ [given from equation (2)].

Step 2: For all $r = 0,1,\dots,N-1$ with $z = h/2$ *and* h:

i. Evaluate $C_i^r(z)$ for each $i = 0,1,\dots,n$. by equation (19).

ii. Compute $\mathcal{A}_r^z$ by equation (20).

Step 3: For each $\ell = 0,1,\dots,m$ and $k = 0,1,\dots,r$ calculate:

$\mathcal{D}[\ell,k] = \left[{}_a^C D_s^{\beta_\ell} u(s)\right]_{s=t_k}$; $\mathcal{D}[\ell,k+\theta_1] = \left[{}_a^C D_s^{\beta_\ell} u(s)\right]_{s=t_{k+\theta_1}}$ and

$\mathcal{D}[\ell,k+\theta_2] = \left[{}_a^C D_s^{\beta_\ell} u(s)\right]_{s=t_{k+\theta_2}}$ from the propositions (2 and 3).

Step 4: For each $r = 0,1,\dots,N-1$ using step (2,i) and step (3) to compute the following:

i. $$\mathcal{B}_1[r] = -\sum_{j=0}^{r-1}\left\{\left[u(t_j+\theta_1 h)-u(t_j)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_{3(r-j)}^{\alpha_i}\right)\right.$$
$$+\left[u(t_j+\theta_2 h)-u(t_j+\theta_1 h)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_{3(r-j)-1}^{\alpha_i}\right)$$
$$\left.+\left[u(t_{j+1})-u(t_j+\theta_2 h)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_{3(r-j)-2}^{\alpha_i}\right)\right\}\text{with } \mathcal{B}_1[0]=0$$

ii. $$\mathcal{R}_1[r] = \frac{\lambda h}{2}\sum_{\ell=0}^{m}\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_r\right)\mathcal{H}_\ell(t_r,\mathcal{D}[\ell,r])$$

Step 5: Calculate for each $r = 0,1,\dots,N-1$

i. $$u(t_r+\theta_1 h) = \frac{1}{\mathcal{A}_r^{\theta_1 h}+C_0^r(\theta_1 h)}\left\{\mathcal{A}_r^{\theta_1 h}u(t_r)+\ \mathcal{B}_1[r]+\mathcal{R}_1[r]+\mathcal{F}(t_r+\ \tfrac{h}{2})\right\}.$$

ii. $$\mathcal{B}_2[r] = -[u(t_r+\theta_1 h)-u(t_r)]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_1^{\alpha_i}\right)$$
$$-\sum_{j=0}^{r-1}\left\{\left[u(t_j+\theta_1 h)-u(t_j)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_{3(r-j)+1}^{\alpha_i}\right)\right.$$
$$+\left[u(t_j+\theta_2 h)-u(t_j+\theta_1 h)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_{3(r-j)}^{\alpha_i}\right)$$
$$\left.+\left[u(t_{j+1})-u(t_j+\theta_2 h)\right]\left(\sum_{i=1}^{n} C_i^r\left(\frac{h}{2}\right) b_{3(r-j)-1}^{\alpha_i}\right)\right\}$$

iii. $$\mathcal{R}_2[r] = \frac{\lambda h}{2}\sum_{\ell=0}^{m}\mathcal{K}_\ell\left(t_r+\frac{h}{2},t_r+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_r+\frac{h}{2},\mathcal{D}[\ell,r+\theta_1]\right)$$

iv. $$u(t_r+\theta_2 h) = \frac{1}{\mathcal{A}_r^{\theta_2 h}+C_0^r(\theta_2 h)}\left\{\mathcal{A}_r^{\theta_2 h}u(t_r+\theta_1 h)+\ \mathcal{B}_2[r]+\mathcal{R}_2[r]+\mathcal{F}\left(t_r\ +\frac{h}{2}\right)\right\}$$

v. $$\mathcal{B}_3[r] = -[u(t_r+\theta_1 h)-u(t_r)]\left(\sum_{i=1}^{n} C_i^r(h)\ b_2^{\alpha_i}\right)-[u(t_r+\theta_2 h)-u(t_r+\theta_1 h)]\left(\sum_{i=1}^{n} C_i^r(h)\ b_1^{\alpha_i}\right)$$
$$-\sum_{j=0}^{r-1}\left\{\left[u(t_j+\theta_1 h)-u(t_j)\right]\left(\sum_{i=1}^{n} C_i^r(h)\ b_{3(r-j)+2}^{\alpha_i}\right)\right.$$
$$+\left[u(t_j+\theta_2 h)-u(t_j+\theta_1 h)\right]\left(\sum_{i=1}^{n} C_i^r(h)\ b_{3(r-j)+1}^{\alpha_i}\right)$$
$$\left.+\left[u(t_{j+1})-u(t_j+\theta_2 h)\right]\left(\sum_{i=1}^{n} C_i^r(h)\ b_{3(r-j)}^{\alpha_i}\right)\right\}$$

vi. $$\mathcal{R}_3[r] = \lambda h\sum_{\ell=0}^{m}\mathcal{K}_\ell\left(t_{r+1},t_r+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_r+\frac{h}{2},\mathcal{D}[\ell,r+\theta_2]\right)$$

Step 6: Compute $u_{r+1} = u(t_{r+1})$ for each $r = 0,1,\dots,N-1$, from:
$$\{\mathcal{A}_r^h + C_0^r(h)\}\,u(t_{r+1}) = \mathcal{A}_r^h\,u(t_r+\theta_2 h)+\ \mathcal{B}_3[r]+\mathcal{R}_3[r]+\mathcal{F}(t_{r+1})$$

where

$$\mathcal{F}(t) = f(t) + \frac{\lambda h}{6}\sum_{\ell=0}^{m}\sum_{k=0}^{r-1}\Big\{\mathcal{K}_\ell(t,t_k)\,\mathcal{H}_\ell(t_k\,,\mathcal{D}[\ell,k]\,) + 2\mathcal{K}_\ell\left(t,t_k+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_k+\frac{h}{2}\,,\mathcal{D}[\ell,k+\theta_1]\,\right)$$
$$+\,2\mathcal{K}_\ell\left(t,t_k+\frac{h}{2}\right)\mathcal{H}_\ell\left(t_k+\frac{h}{2},\mathcal{D}[\ell,k+\theta_2]\right) + \mathcal{K}_\ell(t,t_{k+1})\,\mathcal{H}_\ell(t_{k+1},\mathcal{D}[\ell,k+1])\Big\}$$

With putting $b_\xi^\alpha = (\xi+1)^{1-\alpha} - \xi^{1-\alpha}\ ;\ \xi = 0,1,2,\cdots$

Numerical Examples:

Here, some numerical examples were presented for nonlinear IFDE of V-H type with variable coefficients using Runge-Kutta method (order two, three and four). Their results are obtained by applying previous algorithms with running programs.

Example (1): We first consider a higher-order nonlinear IFDE of V-H type with variable coefficients, for fractional order lies in (0,1):

$$ {}_0^CD_t^{0.9}u(t) + 3\,{}_0^CD_t^{0.6}u(t) + \sinh(t)\,u(t)$$
$$= f(t) + \int_0^t \left\{\frac{s^2e^t}{16}\left[{}_0^CD_s^{0.75}u(s)\right]^4 + (st^2-1)\left[{}_0^CD_s^{0.5}u(s)\right]^2 + e^{s+t}u(s)\right\}ds$$

where

$$f(t) = \frac{-4}{\Gamma(2.1)}t^{1.1} - \frac{12}{\Gamma(2.4)}t^{1.4} + (1-2t^2)\sinh(t) - \frac{2e^t}{\Gamma^4(2.25)}\,t^8 - \frac{16}{\Gamma^2(2.5)}\left\{\frac{t^7}{5}-\frac{t^4}{4}\right\} - 3e^t + 3e^{2t}$$
$$-\,4te^{2t} + 2t^2e^{2t}$$

Together with initial condition: $u(0) = 1\ ;\ 0 \le t \le 1$. While the exact solution is $u(t) = 1 - 2t^2$.

Take $N = 10\,, h = 0.1$ and $t_r = a + rh\ ,\ r = 0,1,2,\ldots,N$.

- **RK2:** from equations (9) and (10), where $n = 2$, for all $r = 0,1,\cdots,N-1$ the following is obtained:
  $\mathcal{A}_r = 21.810204002\ ;\ C_1^r = 13.4607259769; C_2^r = 8.34947802518$
- **RK3:** from equations (14) and (15),where $n = 2$, for all $r = 0,1,\cdots,N-1$ the following is obtained:
  $\mathcal{A}_r(z) = 35.98332227$
  $C_1^r(z) = 20.4026453604\ ; C_2^r(z) = 15.5806769192$ for $z = h/2$ and h
- **RK4:** from equations (19) and (20), where $n = 2$,for all $r = 0,1,\cdots,N-1$ the following is obtained:
  $\mathcal{A}_r(z) = 48.4643901859$
  $C_1^r(z) = 26.0220337702\ ; C_2^r(z) = 22.4423564157$ for $z = h/2$ and h

Table-1: Contain the values of $C_0^r(z)$ for all $r = 0,1,\cdots,N-1;\ z = h/2\,,h$ for R-K method of order two, three, and four:

r	$C_0^r(z)$ in RK2	$C_0^r(z)$ in RK3 & RK4	
	$z = h$	$z = h/2$	$z = h$
0	0.100166750	0.0500208359	0.1001667500
1	0.2013360025	0.1505631331	0.2013360025
2	0.3045202934	0.2526123168	0.3045202934
3	0.4107523258	0.3571897294	0.4107523258
4	0.5210953054	0.4653420169	0.5210953054
5	0.6366535821	0.5781516037	0.6366535821
6	0.7585837018	0.6967475261	0.7585837018
7	0.8881059821	0.8223167319	0.8881059821
8	1.0265167257	0.9561159599	1.0265167257
9	1.1752011936	1.0994843179	1.1752011936

Table-2: presents a comparison between the exact solution and numerical solution by Runge-Kutta methods (order 2,3and 4), also including least square error and running time.

t	Exact solution	Numerical Solution		
		RK2	RK3	RK4
0	1.0	1.0	1.0	1.0
0.1	0.98	0.968284098284	0.97469760863	0.976983906805
0.2	0.92	0.899042545557	0.910621060864	0.914700951482
0.3	0.82	0.791214318239	0.806940721636	0.812693176454
0.4	0.68	0.644319498355	0.66334276569	0.670803604843
0.5	0.5	0.457966574316	0.479658471165	0.488921519461
0.6	0.28	0.231600487601	0.255842470767	0.26692612808
0.7	0.02	−0.03578768713	−0.0079300078881	0.00463995100652
0.8	−0.28	−0.345981594717	−0.31093779951	−0.298239309545
0.9	−0.62	−0.70141372106	−0.650837847666	−0.642287846988
1	−1.0	−1.10301743139	−1.01977486336	−1.02904048148
	$L.S.E$	3.14941346 $e-002$	4.64070292 $e-003$	2.37750015 $e-003$
	$R.Time/Sec$	8.757339	31.301046	70.615406

Table-3: shows the least square errors and running times (elapsed time) for Runge-Kutta Methods (order 2, 3 and 4) with different values of steps size h .

h	$\mathbf{0.1}$ ($\boldsymbol{N=10}$)		0.02 ($\boldsymbol{N=25}$)		0.01($\boldsymbol{N=50}$)	
R-K M	$L.S.E$	$R.Time/Sec$	$L.S.E$	$R.Time/Sec$	$L.S.E$	$R.Time/Sec$
RK2	3.14941346 $e-002$	8.75733	7.7069411 $e-003$	53.76878	2.833247 $e-003$	222.01782
RK3	4.64070292 $e-003$	31.30104	1.3627669 $e-003$	248.66199	5.33190326 $e-004$	1601.4360
RK4	2.37750015 $e-003$	70.61540	5.6207306 $e-004$	746.708480	2.1072918 $e-004$	7651.16816

Example (2): Consider a higher-order nonlinear IFDE of V-H type with variable coefficients on the closed interval[0,1]:

$$ {}_0^C D_t^{2\alpha} u(t) + 3\,{}_0^C D_t^{\alpha} u(t) + (1+t^2)u(t) = f(t) + \int_0^t \left\{ 2st^2 e^{\left\{ s^{2\beta}\,{}_0^C D_s^{2\beta} u(s) \right\}} + (1+st^2)\left\{ {}_0^C D_s^{\beta} u(s) \right\}^2 \right\} ds $$

Where

$$ f(t) = \frac{2}{\Gamma(3-2\alpha)} t^{2(1-\alpha)} + \frac{6}{\Gamma(3-\alpha)} t^{2-\alpha} + t^4 - 1 - \frac{\Gamma(3-2\beta)}{2} t^2 \left\{ e^{\frac{2}{\Gamma(3-2\beta)} t^2} - 1 \right\} - \frac{4}{\Gamma^2(3-\beta)} \left\{ \frac{t^{5-2\beta}}{5-2\beta} + \frac{t^{8-2\beta}}{6-2\beta} \right\} $$

With initial condition:

If $0 < \alpha \; and \; \beta < 0.5$ then $u(0) = -1$.

If $0.5 < \alpha \; or \; \beta < 1$ then $u(0) = -1$, $u'(0) = 0$.

While the exact solution is $u(t) = t^2 - 1$

Take $\boldsymbol{\alpha = 0.4}$, $\boldsymbol{\beta = 0.3}$, $N = 10$, $h = 0.1$ and $t_r = a + rh$, $r = 0,1,2,\dots,N$:

- **RK2:** from equations (9) and (10), where $n = 2$, for all $r = 0,1,\cdots,N-1$ the following is obtained:

$$\mathcal{A}_r = 15.30563167;\ C_1^r = 8.43372114484; C_2^r = 6.8719105251$$

- **RK3:** from equations (14) and (15),where $n = 2$, for all $r = 0,1,\cdots,N-1$ the following is obtained:

$$\mathcal{A}_r(z) = 23.0930529251$$

$$C_1^r(z) = 11.1283617679\ ; C_2^r(z) = 11.9646911573 \text{ For } z = h/2 \text{ and } h$$

- **RK4:** from equations (19) and (20), where $n = 2$,for all $r = 0,1,\cdots,N-1$ the following is obtained:

$$\mathcal{A}_r(z) = 29.63693739207$$

$$C_1^r(z) = 13.0878328303\ ; C_2^r(z) = 16.5491045618 \text{ For } z = h/2 \text{ and } h$$

Table-4: Containing all value of $C_0^r(z)$ for all $r = 0,1,\cdots,N-1,$ and $z = h\ ,h/2$ for R-K method of order two, three and four respectively.

r	$C_0^r(z)$ in RK2	$C_0^r(z)$ in RK3 & RK4	
	$z = h$	$z = h/2$	$z = h$
0	1.01	1.0025	1.01
1	1.04	1.0225	1.04
2	1.09	1.0625	1.09
3	1.16	1.1225	1.16
4	1.25	1.2025	1.25
5	1.36	1.3025	1.36
6	1.49	1.4225	1.49
7	1.64	1.5625	1.64
8	1.81	1.7225	1.81
9	2.0	1.9025	2.0

Table-5: presents a comparison between the exact solution and numerical solution by Runge-Kutta methods (order 2, 3and 4), also including least square error and running time.

t	Exact solution	Numerical Solution		
		RK2	RK3	RK4
0	-1.0	-1.0	-1.0	-1.0
0.1	-0.99	-0.985906928316	-0.988114570591	-0.989135623697
0.2	-0.96	-0.953278173894	-0.957014247459	-0.958708991471
0.3	-0.91	-0.901398343624	-0.906178815421	-0.90842351806
0.4	-0.84	-0.830042502245	-0.835448226443	-0.838216958632
0.5	-0.75	-0.739163195056	-0.744723948559	-0.74806796336
0.6	-0.64	-0.628837037082	-0.633906867107	-0.637972968387
0.7	-0.51	-0.499299567417	-0.502866673981	-0.507944698682
0.8	-0.36	-0.351073928681	-0.351421253488	-0.358021458944
0.9	-0.19	-0.185303990284	-0.179347220279	-0.188292406412
1	0.0	-0.0046697270716	0.013464384070	0.001032871815
	$\boldsymbol{L.S.E}$	$7.15157383\ e-004$	$5.320032\ e-004$	$2.804159e-005$
	$\boldsymbol{R.Time/Sec}$	6.180762	22.500890	54.038023

Table-6: shows the least square errors and running times (elapsed time) for Runge-Kutta Methods (order 2, 3 and 4) with different values of steps size h .

h	$\mathbf{0.1}$ $(N = 10)$		$\mathbf{0.02}$ $(N = 25)$		$\mathbf{0.01}(N = 50)$	
R-KM	$L.S.E$	$R.Time/Sec$	$L.S.E$	$R.Time/Sec$	$L.S.E$	$R.Time/Sec$
RK2	7.15157383 $e-004$	6.18076	2.19005 $e-004$	37.51380	8.7455274 $e-005$	153.6725
RK3	5.320032 $e-004$	22.50089	8.34079 $e-005$	190.2310	2.398982 $e-005$	1354.556
RK4	2.804159 $e-005$	54.03802	8.980861 $e-006$	628.65185	3.7993204 $e-006$	6846.8866

Example (3): Consider a higher-order nonlinear IFDE of V-H type with variable coefficients:

$$ {}_0^C D_t^{\alpha} u(t) + 2(\cos(t) - \sin(t)) {}_0^C D_t^{\alpha - 0.2} u(t) + e^t u(t) $$
$$ = f(t) + \int_0^t \left\{ \Gamma^3(3-\beta)(s^2 - t)\left[{}_0^C D_s^{\beta} u(s)\right]^3 + \frac{se^t}{4}\left[{}_0^C D_s^{\beta-0.2} u(s)\right]^2 + \sin\left(\frac{1}{3}s^3 - s + t\right)u(s)\right\} ds $$

where

$$ f(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + \frac{4}{\Gamma(3.2-\alpha)}(\cos(t) - \sin(t)) t^{2.2-\alpha} + e^t(t^2 - 1) - 8\left\{\frac{t^{-3\beta+9}}{-3\beta+9} - \frac{t^{-3\beta+8}}{-3\beta+7}\right\} $$
$$ - \frac{t^{6.4-2\beta} e^t}{(6.4-2\beta)\Gamma^2(3.2-\beta)} + \cos\left(\frac{t^3}{3}\right) - \cos(t) $$

while the exact solution is $u(t) = t^2 - 1$, together with initial condition:

If $0.2 < \alpha \; and \; \beta < 1$ then $u(0) = -1$

If $1.2 < \alpha \; or \; \beta < 2$ then $u(0) = -1$, $u'(0) = 0$

Take $\boldsymbol{\alpha = 0.65}$, $\boldsymbol{\beta = 0.45}$, $N = 10$, $h = 0.1$ and $t_r = a + rh,\ r = 0,1,2,\ldots,N$.

- **RK2**: from equations (9 and 10), where $n = 2$, for all $r = 0,1,\cdots,N-1$ the following is obtained: $C_2^r = 5.0124319064$
- **RK3**: from equations (14 and 15),where $n = 2$, for all $r = 0,1,\cdots,N-1$ the following is obtained: $C_2^r(z) = 7.86534873123$ for $z = h$ and $h/2$
- **RK4**: from equations (19 and 20),where $n = 2$,for all $r = 0,1,\cdots,N-1$ the following is obtained: $C_2^r(z) = 10.2371083848$ for $z = h$ and $h/2$

Table-7: Containing all value of C_0^r , C_1^r and $\mathcal{A}_r$ for all $r = 0,1,\cdots,N-1$, for R-K-2 method.

r	C_0^r	C_1^r	$\mathcal{A}_r$
0	1.1051709180	5.6767325886	10.6891644951
1	1.2214027581	4.9552370023	9.9676689087
2	1.3498588075	4.1842303258	9.1966622322
3	1.4918246976	3.3714162030	8.3838481094
4	1.6487212707	2.5249160039	7.5373479104
5	1.8221188003	1.6531876788	6.6656195853
6	2.0137527074	0.7649412488	5.7773731553
7	2.2255409284	−0.1309482211	4.8814836852
8	2.4596031111	−1.0255292999	3.9869026065
9	2.7182818284	−1.9098636288	3.1025682776

Table-8: Containing all value of $C_0^r(z)$, $C_1^r(z)$ and $\mathcal{A}_r(z)$ for all $r = 0,1,\cdots,N-1$, for R-K-3 method and $= h/2$.

r	$C_0^r\left(\frac{h}{2}\right)$	$C_1^r\left(\frac{h}{2}\right)$	$\mathcal{A}_r\left(\frac{h}{2}\right)$
0	1.05127109638	8.21897189971	16.08432063094
1	1.16183424273	7.27093601164	15.136284742873
2	1.28402541669	6.25025133439	14.11560006562
3	1.41906754859	5.16711621187	13.03246494310
4	1.56831218549	4.03235297218	11.897701703409
5	1.73325301787	2.85729979451	10.72264852573

6	1.91554082901	1.65369742179	9.519046153019
7	2.11700001661	0.433571851069	8.2989205823007
8	2.33964685193	−0.790885826265	7.074462904966
9	2.58570965932	−2.00744123386	5.8579074973756

Table-9: Containing all value of $C_0^r(z)$, $C_1^r(z)$ and $\mathcal{A}_r(z)$ for all $r = 0,1,\cdots,N-1$, for R-K-3 method and $z = h$.

r	$C_0^r(h)$	$C_1^r(h)$	$\mathcal{A}_r(h)$
0	1.1051709180	7.75464524296	15.6199939741
1	1.2214027581	6.76905322692	14.63440195815
2	1.3498588075	5.71582706859	13.5811757998256
3	1.4918246976	4.6054902556	12.47083898682
4	1.6487212707	3.44913690634	11.314485637569
5	1.8221188003	2.25832092124	10.123669652475
6	2.0137527074	1.04494054001	8.91028927123766
7	2.2255409284	−0.178880541695	7.6864681895359
8	2.4596031111	−1.40091430815	6.46443442307679
9	2.7182818284	−2.60895060193	5.25639812930401

Table-10: Containing all value of $C_0^r(z)$, $C_1^r(z)$ and $\mathcal{A}_r(z)$ for all $r = 0,1,\cdots,N-1$, for R-K-4 method and $z = h/2$.

r	$C_0^r\left(\frac{h}{2}\right)$	$C_1^r\left(\frac{h}{2}\right)$	$\mathcal{A}_r\left(\frac{h}{2}\right)$
0	1.05127109638	9.86412488894	20.1012332737062
1	1.16183424273	8.7263251114	18.9634334961606
2	1.28402541669	7.50133477788	17.73844316264
3	1.41906754859	6.20139358687	16.43850197163354
4	1.56831218549	4.83949012105	15.07659850581
5	1.73325301787	3.42923206966	13.666340454426
6	1.91554082901	1.98471026499	12.22181864975357
7	2.11700001661	0.520357891408	10.75746627617039
8	2.33964685193	−0.949193726219	9.28791465854372
9	2.58570965932	−2.40926131389	7.827847070867

Table-11: Containing all value of $C_0^r(z)$, $C_1^r(z)$ and $\mathcal{A}_r(z)$ for all $r = 0,1,\cdots,N-1$, for R-K-4 method and $z = h$.

r	$C_0^r(h)$	$C_1^r(h)$	$\mathcal{A}_r(h)$
0	1.1051709180	9.3068561469	19.5439645316
1	1.2214027581	8.1239828077	18.3610911924
2	1.3498588075	6.8599373177	17.0970457024
3	1.4918246976	5.5273496016	15.7644579864
4	1.6487212707	4.1395344354	14.3766428202
5	1.8221188003	2.7103584095	12.9474667943
6	2.0137527074	1.2541013783	11.4912097631
7	2.2255409284	−0.21468621927	10.0224221654
8	2.4596031111	−1.68132874318	8.55577964158
9	2.7182818284	−3.13117198605	7.10593639871

Table-12; presents a comparison between the exact solution and numerical solution by Runge-Kutta methods (order 2, 3and 4), also including least square error and running time.

t	Exact solution	Numerical Solution		
		RK2	**RK3**	**RK4**
0	-1.0	-1.0	-1.0	-1.0
0.1	-0.99	-0.986979291148	-0.988258207752	-0.9894281888
0.2	-0.96	-0.955092266204	-0.95734057057	-0.9592038859
0.3	-0.91	-0.903727729192	-0.906708958329	-0.9090786387
0.4	-0.84	-0.832665915405	-0.83626865651	-0.8390171075
0.5	-0.75	-0.74180682188	-0.745998891667	-0.7490075080
0.6	-0.64	-0.631148443014	-0.635872558438	-0.6390406527
0.7	-0.51	-0.500861368263	-0.5057736274	-0.5091019679
0.8	-0.36	-0.351441396779	-0.355357602936	-0.3591698772
0.9	-0.19	-0.183943088883	-0.183818750028	-0.1892243858
1	0.0	-0.000368811900	0.0103947821687	0.0007391054
$\boldsymbol{L.S.E}$		$4.6540541\ e-004$	$2.5357834\ e-004$	$7.3245567\ e-006$
$\boldsymbol{R.Time/Sec}$		8.631298	30.254972	71.696136

Table-13: shows the least square errors and running times (elapsed time) for Runge-Kutta Methods (order 2, 3 and 4) with different values of steps size h .

$\boldsymbol{h}$ R-KM	$\mathbf{0.1}$ $(\boldsymbol{N}=\mathbf{10})$		$\mathbf{0.02}$ $(\boldsymbol{N}=\mathbf{25})$		$\mathbf{0.01}(\boldsymbol{N}=\mathbf{50})$	
	$\boldsymbol{L.S.E}$	$\boldsymbol{R.Time/Sec}$	$\boldsymbol{L.S.E}$	$\boldsymbol{R.Time/Sec}$	$\boldsymbol{L.S.E}$	$\boldsymbol{R.Time/Sec}$
RK2	$4.6540541\ e-004$	8.631298	$1.13306\ e-004$	54.9339	$3.720853\ e-005$	223.45131
RK3	$2.5357834\ e-004$	30.254972	$3.918257\ e-005$	245.50199	$9.7522997\ e-006$	1899.20312
RK4	$7.3245567\ e-006$	71.696136	$1.66223681\ e-006$	755.4621	$5.69750175\ e-007$	7135.7011

Conclusion:

In general, finding the exact solutions of multi-order nonlinear integro-fractional differential equations (IFDEs) is difficult and needs more mathematical computation or mostly impossible. In this work, the Runge-Kutta method of order two, three and four is generalized and applied to Volterra-Hammerstein type and the order of Caputo derivatives in the range of $(0,1)$, which is a good model for nonlinear IFDE's with non-constant coefficients.

For each order of R-K method, a computer programs (MatLab) was written and several examples were included for illustration. For the comparison of computing accuracy and speed the least square error and running time of the associative program are given in tabular forms, from which the following points are concluded:

1. The numerical experiments have shown that the fourth order Runge-Kutta method is the most popular method that gives the best approximation to the exact solution amongst the other orders.
2. We get a good accuracy if we choose N sufficiently large (small step size).
3. Runge-Kutta method of second order is faster, and lower in accuracy, than the other orders.

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